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Topological conjugacy on the complement of the periodic points

Stephen M. Shea*

Department of Mathematics, St. Anselm College, 100 St. Anselm Drive 1792, Manchester, NH 03102, United States

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Abstract

We will say that two subshifts are essentially conjugate if they are topologically conjugate on the complement of their periodic points. In 1990, Susan Williams presented an example of a sofic shift that is not topologically conjugate to a renewal system. We show that the example of Williams is essentially conjugate to a renewal system. We also present an example of a renewal system that is essentially conjugate to a shift of finite type but not topologically conjugate to a shift of finite type. Finally, we prove that all renewal systems that meet a certain technical condition are essentially conjugate to a shift of finite type.

Keywords: Conjugacy; Renewal system; Shift of finite type; Sofic

1. Introduction

For a thorough introduction to symbolic dynamics, see [7]. Let A be a finite set of states. The full shift $(A^{\mathbb{Z}}, \sigma)$ is the set of all bi-infinite sequences of symbols in A together with the shift map σ, where, if $x \in A^{\mathbb{Z}}$, $(\sigma(x))_n = x_{n+1}$. A subshift (or shift space) (X, σ) is a closed shift invariant subset of the full shift. We say that two shift spaces (X, σ_1) and (Y, σ_2) are topologically conjugate if there exists a homeomorphism $\phi : X \to Y$ such that $\phi\sigma_1 = \sigma_2\phi$. Shifts of finite type (SFTs) are shift spaces that can be described by some finite set of forbidden blocks. A shift

* Tel.: +1 860 301 7442.
E-mail addresses: sshea@anselm.edu, sshea@wesleyan.edu.

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space is sofic if and only if it is a factor of a shift of finite type [7]. A renewal system is a subshift generated by free concatenations of a finite set of words [10].

Let (X, σ_1) be a shift space. A point $x \in X$ is periodic if, for some $n \in \mathbb{N}$, $\sigma_1^n(x) = x$. Let $X^* = \{x \in X | x \text{ is not periodic}\}$.

Definition 1.1. We say that (X, σ_1) and (Y, σ_2) are essentially conjugate if there exists a homeomorphism $\phi : X^* \to Y^*$ such that $\phi\sigma_1 = \sigma_2\phi$.

Although the terminology "essentially conjugate" is new to this article, Definition 1.1 is due to Hochman, and was introduced with the following problem [3].

Problem 1.2 (*Using Our Terminology*)**.** Let X and Y be mixing SFTs on finite alphabets. Is X essentially conjugate to Y?

Here, we do not investigate essential conjugacies between shifts of finite type, but rather look at essential conjugacies of renewal systems. There are still important open conjugacy problems regarding renewal systems and SFTs. For example, we do not know if every irreducible SFT is topologically conjugate to a renewal system [4]. The apparent difficulty of this and related conjugacy problems yields more motivation for studying essential conjugacies. An essential conjugacy is a finitary code. In Section 2, we provide some background on related finitary theories. In Section 3, we present two examples of non-conjugate systems that are essentially conjugate. In Section 4, we prove that all renewal systems that meet a certain technical condition are essentially conjugate to an SFT.

2. Finitary codes

We know of three interesting theories that are related to the theory of essential conjugacies. In [3], Hochman does a nice job of relating essential conjugacies to two of these: the almost isomorphism theory of Adler and Marcus [1] and the theory of entropy conjugacy defined by Buzzi [2]. So, we will focus on the third related theory, Keane and Smorodinsky's work on finitary isomorphisms of Bernoulli schemes [5].

A process is a quadruple $(X, \mathcal{U}, \mu, T)$, where X is the set of doubly infinite sequences on some alphabet A, $\mathcal{U}$ is the σ-algebra generated by the coordinates, μ is a shift invariant probability measure on $(X, \mathcal{U})$, and T is the left shift by one. A process is a Bernoulli scheme if $\mu = p^{\mathbb{Z}}$ for some probability vector p. Let $(X, \mathcal{U}, \mu, T)$ and $(Y, \mathcal{V}, \nu, S)$ be two processes. An isomorphism ϕ from $(X, \mathcal{U}, \mu, T)$ to $(Y, \mathcal{V}, \nu, S)$ is a bimeasurable equivariant map from a subset of X of measure one to a subset of Y of measure one which takes μ to ν.

Ornstein proved that entropy is a complete isomorphism invariant for Bernoulli schemes [9]. Prior to Ornstein's result, mathematicians began trying to construct isomorphisms between various Bernoulli schemes. In 1959, Meshalkin showed that Bernoulli schemes with non-isomorphic state spaces could be isomorphic [8]. Meshalkin constructed not only an isomorphism, but a finitary isomorphism. Finitary isomorphism is formally defined as follows [5]. We will use the shorthand $x[m, n]$ to mean $x_m x_{m+1} \dots x_n$.

Definition 2.1. An isomorphism ϕ from $(X, \mathcal{U}, \mu, T)$ to $(Y, \mathcal{V}, \nu, S)$ is finitary if for almost every $x \in X$ there exist integers $m \le n$ such that the zero coordinates of $\phi(x)$ and $\phi(x')$ agree for almost all $x' \in X$ with $x[m, n] = x'[m, n]$, and similarly for ϕ^{-1}.

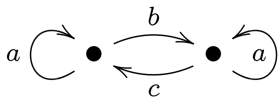

Fig. 1. Williams' example.

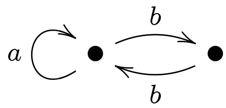

Fig. 2. Even shift.

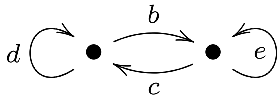

Fig. 3. Y of Example 3.1.

Keane and Smorodinsky proved that entropy is a complete finitary isomorphism invariant for Bernoulli schemes [5]. Later, they showed that irreducible, finite-state Markov processes are finitarily isomorphic if and only if they have the same entropy and period [6]. Let X and Y be two equal topological entropy-irreducible SFTs. If we assign X and Y equal-entropy, shift-invariant, aperiodic Markov measures, there exists a finitary isomorphism between the resulting Markov processes. We emphasize that this finitary isomorphism does not necessarily define an essential conjugacy between the SFTs. The complement of the periodic points of X, denoted X^*, is a subset of full measure, but a subset of full measure need not include all points in X^*. For example, if X is the full shift on two symbols a and b, then the finitary isomorphism need not be defined on the point x that is an infinite string of a's followed by an infinite string of b's. In fact, well-known methods for constructing finitary isomorphisms such as the marker and filler methods used by Keane and Smorodinsky in [5,6] are not necessarily defined on this point x.

3. Two examples of essential conjugacy

In [10], Susan Williams presented the sofic shift given by the directed graph in Fig. 1 as an example of a sofic shift that is not topologically conjugate to a renewal system. In [4], the renewal system given by the directed graph in Fig. 2 is presented as an example of a renewal system that is not topologically conjugate to a shift of finite type. In this section, we prove that the sofic shift given by the directed graph in Fig. 1 is essentially conjugate to a renewal system, and the renewal system given by the graph in Fig. 2 is essentially conjugate to a shift of finite type.

Example 3.1. The sofic shift (X, σ_1) given by the directed graph in Fig. 1 is essentially conjugate to a renewal system.

Proof. Let Y be the shift of finite type given by the directed graph in Fig. 3.

Let $\phi : X^* \to Y^*$ be defined as follows.

(i) $(\phi(x))_n = x_n$ if $x_n = b$ or c.

(ii) $(\phi(x))_n = d$ if there exists an $m < n$ such that $x_i = a$ for $m < i \leq n$ and $x_m = c$, or there exists $p > n$ such that $x_i = a$ for $n \leq i < p$ and $x_p = b$.

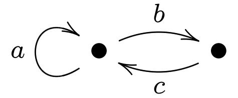

Fig. 4. Y of Example 3.2.

(iii) $(\phi(x))_n = e$ if there exists an $m < n$ such that $x_i = a$ for $m < i \leq n$ and $x_m = b$, or there exists $p > n$ such that $x_i = a$ for $n \leq i < p$ and $x_p = c$.

Since $a^\infty \notin X^*$, and a^∞ is the only point in X for which ϕ is not defined, ϕ is defined on X^*. The inverse of ϕ is simply a one-block code that fixes b and c and sends both d and e to a. Thus ϕ^{-1} is defined on all points of Y. The maps ϕ and ϕ^{-1} are continuous on X^* and Y^*, respectively, and commute with the shift.

Let $\psi : Y \to Z$ be defined as follows.

(i) $(\psi(y))_n = d$ if $y_n = c$ or d, and
(ii) $(\psi(y))_n = e$ if $y_n = b$ or e.

Then ψ^{-1} is a two-block code, where

(i) $(\psi^{-1}(z))_n = b$ if $z_n = e$ and $z_{n-1} = d$,
(ii) $(\psi^{-1}(z))_n = c$ if $z_n = d$ and $z_{n-1} = e$,
(iii) $(\psi^{-1}(z))_n = d$ if $z_n = d$ and $z_{n-1} = d$, and
(iv) $(\psi^{-1}(z))_n = e$ if $z_n = e$ and $z_{n-1} = e$.

The map ψ is a topological conjugacy. Z is a renewal system generated by $\{d, e\}$. Since ϕ is an essential conjugacy and ψ is a topological conjugacy, $\psi(\phi) : X^* \to Z^*$ is an essential conjugacy from X to a renewal system. $\square$

Our next result also follows from Theorem 4.5 of the next section. We present the proof of this special case, because it is far less technical than the proof for the general case.

Example 3.2. The renewal system (X, σ_1) given by the graph in Fig. 2 is essentially conjugate to a shift of finite type.

Proof. Let Y be the SFT given by the directed graph in Fig. 4.

If $x \in X^*$ and $x_n = b$, then there exists a nonnegative integer d such that $x_i = b$ for $n - d \leq i \leq n + d$, and either $x_{n-d-1} = a$ or $x_{n+d+1} = a$. This is because $b^\infty \notin X^*$, and X is generated by $\{a, bb\}$.

In the following definition, let d be as defined above. Define $\phi : X^* \to Y^*$ as follows.

(i) $(\phi(x))_n = x_n$ if $x_n = a$,
(ii) $(\phi(x))_n = c$ if $x_n = b$, d is odd, and $x_{n-d-1} = a$,
(iii) $(\phi(x))_n = c$ if $x_n = b$, d is even, and $x_{n+d+1} = a$, and
(iv) $(\phi(x))_n = b$ otherwise.

The inverse of ϕ is a one-block code that fixes a and b and sends c to b. The maps ϕ and ϕ^{-1} are continuous on X^* and Y^*, respectively, and commute with the shift. $\square$

4. Main result

Our main result will be that renewal systems that meet a certain technical condition are essentially conjugate to an SFT. If a renewal system X has a generating set where no symbol occurs more than once in any generating word and no symbol occurs in more than one generating word, then X is an SFT. We have already seen in Example 3.2 that a renewal system where one symbol occurs twice in the same generating word can be essentially conjugate to an SFT. We will show that it is also not necessarily a problem to have the same symbol occur in multiple words in the generating set. Having the same symbol occur multiple times in a generating set opens the possibility for generating words to overlap with each other (e.g. 123 overlaps with 234). For our methods, certain types of overlap will be problematic. The next few definitions will give us the terminology to talk about problematic overlap. Informally, our technical condition in Definition 4.3 will be that the generating set has only finitely many problematic overlaps. We will present several examples throughout this section to help motivate and explain our definitions.

Let W be some finite set of words. Throughout this section, we let $W^{\#}$ denote the set of concatenations of words from W and W^{∞} denote the set of bi-infinite sequences that are concatenations of words from W. We will also let $|w|$ be the length of the word w. Let X be a renewal system. W is a generating set for X if $X = W^{\infty}$. We will denote words with lowercase letters and sets of words with uppercase letters. For the rest of this section, let W be a generating set for X. We can assume that the words in W are distinct. Let $k \in \{2, 3, 4, \ldots\}$. We can assume that there do not exist $w_1, w_2, \ldots, w_k$ in W such that there exists $w \in W$, where $w = w_1 w_2 \ldots w_k$. For $w, v \in W^{\#}$, we write $w \subset v$ if w is a subword of v. We will need the following stronger relationship between words in $W^{\#}$.

Definition 4.1. Let $k, l \in \mathbb{Z}^+$. Let $w = w_1 w_2 \ldots w_k$ and $v = v_1 v_2 \ldots v_l$, where $w_i, v_j \in W$ for $1 \le i \le k$ and $1 \le j \le l$. Suppose that $w \subset v$. We write $w \Subset v$ if there exist integers $0 \le m_1 \le m_2 \le \cdots \le m_{k+1} \le n + 1$ such that, for some $x \in W^{\infty}$, the following hold.

(i) $x[0, n] = v$,
(ii) $x[m_r, m_{r+1} - 1] = w_r$ for $1 \le r \le k$,
(iii) $m_1 < |v_1|$, $m_{k+1} - 1 > n - |v_l|$, and
(iv) if for some s where $1 \le s \le k$ and some t, where $1 \le t \le l$, $m_r = (\sum_{u=1}^{s} |v_u|) - |v_s|$, then $m_{r+1} \ne \sum_{u=1}^{s} |v_u|$.

Parts (iii) and (iv) of Definition 4.1 are what make $\Subset$ stronger than $\subset$. Then, parts (i) and (ii) may seem like an unnecessarily formal way of restating that $w \subset v$. However, parts (i) and (ii) are needed to set up parts (iii) and (iv). To explain part (iv), suppose that we have (abc) and $(bc) \in W$. We know that $(bc) \subset (abc)$. We also have that $(bc) \Subset (abc)$. We know that $(bc)(bc) \subset (abc)(bc)$. However, because of part (iv), $(bc)(bc) \not\Subset (abc)(bc)$. Also note that, for any word $w \in W^{\#}$, $w \not\Subset w$. By part (iii), $(bc) \not\Subset (abc)(bc)$.

Recall that, in Example 3.2, $\{a, bb\}$ generated X. In this case, $(bb) \Subset (bb)(bb)$, since to satisfy Definition 4.1 we can take $x[0, 3] = bbbb$ and take $m_1 = 1$.

Let $w \in W^{\#}$. A parsing of w is a concatenation of generating words equal to w. It is possible that a word may have more than one parsing. For example, let $W = \{1234, 56, 123, 456\}$. Then $123456 = (123)(456) = (1234)(56)$.

Definition 4.2. We say that $w \in W^{\#}$ can be parsed periodically if there exists $v \in W^{\#}$ and an integer $n > 1$ such that $w = v^n$.

Let $W^* = \{w \in W^{\#} | w \text{ cannot be parsed periodically}\}$.

Definition 4.3. Let $w \in W$. If there exists (a possibly finite) sequence $w_1, w_2, \ldots \in W^\#$, where $w_1 \in W^*$ and $w \Subset w_1 \Subset w_2 \Subset \cdots$,, then we say that $w_1, w_2, \ldots$ is an extension of w. We say that W is bounded by a positive integer M if, for all $w \in W$ where there exists v_1, v_2 in the extension of w such that $|v_2| > M$, and $v_1 \Subset v_2$, then there exists $v \in W$ and a positive integer n such that $v_1 = v^n$ and $v_2 = v^{n+1}$.

Note that, if W is bounded, then, for any $w \in W$, there are only finitely many $v \in W^*$ such that v is in an extension of w.

Let us give an example of a generating set that is not bounded. Consider the renewal system X generated by $W = \{12, 21, 23, 11\}$. We can construct the sequence $21 \Subset (12)(11) \Subset (11)(21)(11) \Subset (11)(12)(11)(11) \Subset \cdots$, Since this sequence lives strictly in W^* and does not terminate, W is not bounded. In our example, 23 cannot follow 321^K unless K is odd. It is a simple exercise to show that this implies X is not topologically conjugate to a shift of finite type. We do not know if X is essentially conjugate to a shift of finite type.

When constructing the essential conjugacy in Example 3.2, we relabeled one of the b's with a c. To do this, we needed to always (in X^*) be able to determine which b we were considering. In general, when constructing an essential conjugacy from a renewal system to an SFT, we will want to replace duplicate appearances of a symbol in the generating set with a new unique symbol. Consider again our example of a renewal system X generated by $W = \{12, 21, 23, 11\}$. Consider the point $x = \cdots 1112111 \cdots$. Clearly, $x \in X^*$. However, we have no way of determining if that one 2 came from the word (21) or (12). This is problematic for our methods. Thus, we impose the technical condition that our generating sets are bounded. If, for example, we let $W = \{21, 11\}$, then W is bounded. We need one more definition for the proof of our main theorem.

Definition 4.4. Let W be a bounded generating set. Let $V \subset W^\#$. A word $v \in V$ where $|v| < M$ is maximal in V if, for some $w \in W$, v is in an extension of w and, if $v_1 \in V$ such that $v \Subset v_1$, then there exists $u \in W$ and a positive integer n such that $v = u^n$ and $v_1 = u^{n+1}$.

Suppose that $W = \{21, 11\}$. Then $11 \Subset 2111$, and there does not exist $u \in W^\#$ such that $2111 \Subset u$. Therefore, 2111 is maximal in W^*. In Example 3.2, X is generated by $W = \{a, bb\}$. In this case, W is bounded and $W^\#$ has no maximal words. If we let $W = \{21, 12\}$, then again W is bounded, and W does not contain maximal words. If, for some bounded generating set W, there exists $w \in W$, $v \in W^*$ such that $w \Subset v$, then $W^\#$ must contain a maximal word.

We now present our main result.

Theorem 4.5. *If X is a renewal system with bounded generating set W, then X is essentially conjugate to an SFT.*

Proof. There will be four steps to our proof. First, we will define a finite number of conjugacies, the composition of which forms a conjugacy from X to a sofic shift Y. Second, we will define an essential conjugacy from Y to a sofic shift Z. From our work in Steps 1 and 2, it may be clear to the reader that the essential conjugacy is well defined. Still, all of these details are presented in Step 3. In Step 4, we show that Z is an SFT.

Step 1

Suppose that W has minimal bound M. If there does not exist a generating word w and a word $v \in W^*$ such that $w \Subset v$, then let $X = Y$. Otherwise, let w_0 be a maximal word in $W^\#$. We now define a conjugacy $\phi_1 : X \to Y^1$ that relabels w_0 and fixes everything else. Suppose that $w_0 = a_1 a_2 \ldots a_l$, where, for $1 \le m \le l$, a_m is a symbol in the alphabet of X. Note that the a_m's are not necessarily distinct. Let b_{a_m} for $1 \leq m \leq l$ be symbols not in the alphabet of X. Here we emphasize that, if for some i, j where $1 \leq i \leq l$ and $1 \leq j \leq l$, $a_i = a_j$, then $b_{a_i} = b_{a_j}$. For $1 \leq m \leq l$, define $\phi_1 : X \to Y^1$ so that $(\phi_1(x))_p = b_{a_m}$ if $x_p = a_m$ and $x[p-m+1, p+(l-m)] = w_0$. Otherwise, $(\phi_1(x))_p = x_p$.

We claim that Y^1 is a renewal system. Recall that, for some positive integer L, $w_0 = w_1 w_2 \dots w_L$, where, for $1 \leq j \leq L$, $w_j \in W$. Since w_0 is maximal in $W^\#$, if $w_0 \Subset v$, then there exists a $u \in W$ and $n \in \mathbb{N}$ such that $w_0 = u^n$ and $v = u^{n+1}$. Thus, Y^1 is a renewal system generated by concatenations of any (not necessarily distinct) L words in W, except that w_0 is replaced by $\phi_1(w_0)$.

Let $W^\# \setminus w_0 = \{w \in W^\# | w_0 \not\subset w\}$. If in $W^\# \setminus w_0$ there exists a generating word w and a word v that cannot be parsed periodically such that $w \Subset v$, then we repeat the procedure. Let w_1 be a maximal word in $W^\# \setminus w_0$. We then define $\phi_2 : Y^1 \to Y^2$ so that ϕ_2 relabels w_1 with symbols not in the alphabet of Y^1 (in the same fashion as ϕ_1). We continue the procedure. Since W is finite, the set of words in X of length less than M is finite. So, the procedure must eventually stop. This implies that there exists a positive integer K such that Y^K is a renewal system, and such that, for any $u, v, w \in W$ such that $v \neq w$, if $u \Subset v$ then v is not allowable in Y^K and if $u \Subset vw$, then vw is not allowable in Y^K. Call this statement (S1).

Let $Y = Y^K$. Then, for some integer N, Y is generated by $\{v_0, v_1, \dots, v_{K-1}\} \cup \{v_K, v_{K+1}, \dots, v_{K+N}\}$, where, for $0 \leq i \leq K-1$, $v_i = \phi_{i+1}(w_i)$, and for $K \leq i \leq K+N$, $v_i \in W^\#$. Let $W' = \{w \in W | w \subset v_i$ for some i, where $K \leq i \leq K+N\}$.

Let $W'' = W' \cup \{v_0, v_1, \dots, v_{K-1}\}$. Then we have for any three words $u, v, w \in W''$, where $v \neq w$, if $u \Subset v$ then v is not allowable in Y, and if $u \Subset vw$, then vw is not allowable in Y. Denote this by (S2). (S2) holds because the condition holds in W' by (S1), and the generating words in $\{v_0, v_1, \dots, v_{K-1}\}$ have been relabeled so that no symbol occurs in more than one word in $\{v_0, v_1, \dots, v_{K-1}\}$.

We pause here to note that W'' is not a generating set for Y, but that $Y \subset W''^\infty$.

Step 2

It may be the case that some symbol in the alphabet of X occurs in at least two words in W'' or at least twice in any one word in W''. We now define an essential conjugacy $\phi : Y^* \to Z^*$ that uniquely relabels every word in W''. That is, the image of W'' under ϕ will be a set of words $\phi(W'')$ such that no symbol in the alphabet of Z occurs in more than one word in $\phi(W'')$ or more than once in any word in $\phi(W'')$.

Let $k' = \max\{|w|$ such that $\in W''\}$. Let $y \in Y^*$, where $y[1, l] = w$ for some $w \in W''$.

Since w^∞ is not in Y^*, there exist nonnegative integers d, j, j', m', and n', where $w_j \in W''$, $w_{j'} \in W''$, $w_j \neq w$, $w_{j'} \neq w$, $-m' \leq -dl$, and $n' \geq (d+1)l + 1$, such that the following hold.

(i) $y[1 - dl, (d+1)l] = w^{2d+1}$, and either
(iia) $y[-dl - l + 1, -dl] \neq w$ and $y[-m', -dl] = w_j$, or
(iib) $y[(d+1)l + 1, (d+1)l + l] \neq w$ and $y[(d+1)l + 1, n'] = w_{j'}$.

In other words, given that w occurs at $y[1, l]$, if we look far enough in the future and past of y, we will eventually find a word that is not w. It is possible that both $y[(d+1)l + 1, n'] = w_{j'}$ and $y[-m', -dl] = w_j$.

Let $|W''|$ denote the number of words in W''. For $1 \leq i \leq |W''|$ and $1 \leq j \leq k'$, let a_{ij} be symbols not in the alphabet of Y. We now define the essential conjugacy $\phi : Y^* \to Z^*$. For $w_i \in W''$, our essential conjugacy ϕ will relabel w_i with symbols $a_{i1} a_{i2} \dots a_{i|w|}$.

For $w_i \in W''$ and $1 \leq m \leq |w|$, define $\phi : Y^* \to Z^*$ so that $(\phi(y))_p = a_{im}$ if there exist integers d, j, j', m', and n', where $d \geq 0$, $w_j \in W''$, $w_{j'} \in W''$, $w_j \neq w_i$, $w_{j'} \neq w_i$, $m' \leq p - m - dl$, and $n' \geq p + (l - m) + dl + 1$ such that the following hold.

(i) $y[p - m - dl + 1, p + (l - m) + dl] = w_i^{2d+1}$, and either
(iia) $y[p - m - dl - l + 1, p - m - dl] \neq w_i$ and $y[m', p - m - dl] = w_j$, or
(iib) $y[p + (l - m) + dl + 1, p + (l - m) + dl + l] \neq w_i$ and $y[p + (l - m) + dl + 1, n'] = w_{j'}$.

Recall that, in Example 3.2, we had to look forward and back in x to an occurrence of a to determine where to map a b. Here, again, our w_i may be a word such as (bb) (where $w_i \Subset w_i w_i$). Thus, it is necessary that ϕ be an essential conjugacy and not simply a conjugacy.

Step 3
We now check that, for each $y \in Y^*$, there is one unique output z in Z^* under ϕ. We show that, if there is more than one output for a given input, then we contradict (S2). We will have to check many cases, but the same approach is used in each case.

Suppose that there are two outputs for a given input $y \in Y^*$. Then there exist integers p, i, i', m, and n, where $w_i \in W''$, $w_{i'} \in W''$, $1 \leq m \leq |w_i|$, $1 \leq n \leq |w_{i'}|$, and either $i \neq i'$ or $m \neq n$, such that $(\phi(y))_p$ may be a_{im} and $a_{i'n}$. Let $l = |w_i|$ and $l' = |w_{i'}|$. Then there exist integers d, j, j', m', and n', where $d \geq 0$, $w_j \in W''$, $w_{j'} \in W''$, $w_j \neq w_i$, $w_{j'} \neq w_i$, $m' \leq p - m - dl$, and $n' \geq p + (l - m) + dl + 1$ such that the following hold.

(i) $y[p - m - dl + 1, p + (l - m) + dl] = w_i^{2d+1}$, and either
(iia) $y[p - m - dl - l + 1, p - m - dl] \neq w_i$ and $y[m', p - m - dl] = w_j$, or
(iib) $y[p + (l - m) + dl + 1, p + (l - m) + dl + l] \neq w_i$ and $y[p + (l - m) + dl + 1, n'] = w_{j'}$.

Also, there exist integers $\hat{d}$, $\hat{j}$, $\hat{j}'$, $\hat{m}'$, and $\hat{n}'$, where $\hat{d} \geq 0$, $w_{\hat{j}} \in W''$, $w_{\hat{j}'} \in W''$, $w_{\hat{j}} \neq w_{i'}$, $w_{\hat{j}'} \neq w_{i'}$, $\hat{m}' \leq p - n - \hat{d}l'$, and $\hat{n}' \geq p + (l' - n) + \hat{d}l' + 1$ such that the following hold.

(i) $y[p - n - \hat{d}l' + 1, p + (l' - n) + \hat{d}l'] = w_{i'}^{2\hat{d}+1}$, and either
(iia) $y[p - n - \hat{d}l' - l' + 1, p - n - \hat{d}l'] \neq w_{i'}$ and $y[\hat{m}', p - n - \hat{d}l'] = w_{\hat{j}}$, or
(iib) $y[p + (l' - n) + \hat{d}l' + 1, p + (l' - n) + \hat{d}l' + l'] \neq w_{i'}$ and $y[p + (l' - n) + \hat{d}l' + 1, \hat{n}'] = w_{\hat{j}'}$.

We consider three cases, although Cases 2 and 3 follow from Case 1 with minor modifications. Each case will have subcases labeled (A) and (B). Each subcase may have as many as five subcases labeled (a) through (e). Each of these subcases may have as many as four subcases labeled (i) through (iv). Again, in all cases, we find a contradiction to (S2).

Case 1: Suppose that (iia) is satisfied in both instances. That is, $y[m', p - m - dl] = w_j$ and $y[\hat{m}', p - n - \hat{d}'l] = w_{\hat{j}}$.

(A) First, suppose that $i = i'$. Without loss of generality, suppose that $m < n$. Recall that $j \neq i$.
 (a) If $|w_j| \geq n - m$, then $w_{i'} \Subset w_j w_i$.
 (b) Otherwise, $w_j \Subset w_{i'}$.
 In both (a) and (b), we have contradicted (S2).
(B) Now, suppose that $i \neq i'$. Without loss of generality, suppose that $\hat{m}' \leq m'$.
 (a) If $p - m - dl < p - n - \hat{d}l'$, then $w_j \Subset w_{\hat{j}}$.
 (b) (i) If $p - m - dl = p - n - \hat{d}l'$ and $|w_i| < |w_{i'}|$, then $w_i \Subset w_{i'}$.
 (ii) If $p - m - dl = p - n - \hat{d}l'$ and $|w_{i'}| < |w_i|$, then $w_{i'} \Subset w_i$.

(c) Suppose that $p - m - dl > p - n - \hat{d}l'$ and $p - n - \hat{d}l' > m' - 1$.
  (i) If $p - n - \hat{d}l' + |w_{i'}| \leq p - m - dl$, then $w_{i'} \Subset w_j$.
  (ii) If $p - n - \hat{d}l' + |w_{i'}| > p - m - dl$, then $w_j \Subset w_{\hat{j}} w_{i'}$.

(d) Suppose that $p - m - dl > p - n - \hat{d}l'$ and $p - n - \hat{d}l' < m' - 1$.
  (i) If $p - n - \hat{d}l' + |w_{i'}| \geq p - m - dl$, then $w_j \Subset w_{i'}$.
  (ii) If $p - n - \hat{d}l' + |w_{i'}| < p - m - dl$ and $p - n - \hat{d}l' + 2|w_{i'}| \leq p - m - dl + |w_i|$, then $w_{i'} \Subset w_j w_i$.
  (iii) Otherwise, $w_i \Subset w_{i'}$.

(e) Suppose that $p - m - dl > p - n - \hat{d}l'$ and $p - n - \hat{d}l' = m' - 1$.
  (i) If $w_{i'} = w_j$ and $|w_i| < |w_{i'}|$, then $w_i \Subset w_{i'}$.
  (ii) If $w_{i'} = w_j$ and $|w_{i'}| < |w_i|$, then $w_{i'} \Subset w_i$.
  (iii) If $w_{i'} \neq w_j$ and $|w_{i'}| < |w_j|$, then $w_{i'} \Subset w_j$.
  (iv) If $w_{i'} \neq w_j$ and $|w_j| < |w_{i'}|$, then $w_j \Subset w_{i'}$.

Case 2: Suppose that (iib) is satisfied in both instances. That is, $y[p+(l-m)+dl+1, n'] = w_{j'}$ and $y[p + (l' - n) + \hat{d}l' + 1, \hat{n}'] = w_{\hat{j}'}$. There is a symmetry with Case 1. For completeness, we provide the complete proof of Case 2.

(A) First, suppose that $i = i'$. Without loss of generality, suppose that $m < n$.
  (a) If $\hat{n}' \geq p + (l - m) + dl$, then $w_i \Subset w_{i'} w_{\hat{j}'}$.
  (b) Otherwise, $w_{\hat{j}'} \Subset w_i$.

(B) Now, suppose that $i \neq i'$. Without loss of generality, suppose that $\hat{n}' \leq n'$.
  (a) (i) If $p + (l' - n) + \hat{d}l' + 1 = p + (l - m) + dl + 1$ and $|w_{i'}| < |w_i|$, then $w_{i'} \Subset w_i$.
  (ii) If $p + (l' - n) + \hat{d}l' + 1 = p + (l - m) + dl + 1$ and $|w_i| < |w_{i'}|$, then $w_i \Subset w_{i'}$.
  (b) If $p + (l' - n) + \hat{d}l' + 1 > p + (l - m) + dl + 1$, then $w_{\hat{j}'} \Subset w_{j'}$.
  (c) Suppose that $p+(l'-n)+\hat{d}l'+1 < p+(l-m)+dl+1$ and $\hat{n}'+1 > p+(l-m)+dl+1$.
  (i) If $p + (l - m) + dl + 1 - |w_i| \leq p + (l' - n) + \hat{d}l' + 1$, then $w_{\hat{j}'} \Subset w_i w_{j'}$.
  (ii) Otherwise, $w_i \Subset w_{\hat{j}'}$.
  (d) Suppose that $p+(l'-n)+\hat{d}l'+1 < p+(l-m)+dl+1$ and $\hat{n}'+1 < p+(l-m)+dl+1$.
  (i) If $p + (l - m) + dl + 1 - |w_i| \leq p + (l' - n) + \hat{d}l' + 1$, then $w_{\hat{j}'} \Subset w_i$.
  (ii) If $p+(l-m)+dl+1-|w_i| > p+(l'-n)+\hat{d}l'+1$ and $p+(l-m)+dl+1-2|w_i| \geq p + (l' - n) + \hat{d}l' + 1 - |w_{i'}|$, then $w_i \Subset w_{i'} w_{\hat{j}'}$.
  (iii) Otherwise, $w_{i'} \Subset w_i$.
  (e) Suppose that $p+(l'-n)+\hat{d}l'+1 < p+(l-m)+dl+1$ and $\hat{n}'+1 = p+(l-m)+dl+1$.
  (i) If $w_i = w_{\hat{j}'}$ and $|w_{i'}| < |w_i|$, then $w_{i'} \Subset w_i$.
  (ii) If $w_i = w_{\hat{j}'}$ and $|w_i| < |w_{i'}|$, then $w_i \Subset w_{i'}$.
  (iii) If $w_i \neq w_{\hat{j}'}$ and $|w_i| < |w_{\hat{j}'}|$, then $w_i \Subset w_{\hat{j}'}$.
  (iv) If $w_i \neq w_{\hat{j}'}$ and $|w_{\hat{j}'}| < |w_i|$, then $w_{\hat{j}'} \Subset w_i$.

Case 3: Suppose that (iia) is satisfied in one instance and (iib) is satisfied in the other, and that we are not in Case 1 or Case 2. Without loss of generality, suppose that $y[m', p - m - dl] = w_j$ and $y[p + (l' - n) + \hat{d}l' + 1, \hat{n}'] = w_{\hat{j}'}$. Since we are not in Case 1, there exists $\hat{m}'$, where $\hat{m}' \leq p - n - \hat{d}l'$ and $y[\hat{m}', p - n - \hat{d}l'] = w_{i'}$. We now follow the proof in Case 1, replacing $w_{\hat{j}}$ with $w_{i'}$ (and making a minor modification in 3Bc).

(A) First, suppose that $i = i'$. Without loss of generality, suppose that $m < n$.
  (a) If $|w_j| \geq n - m$, then $w_{i'} \Subset w_j w_i$.
  (b) Otherwise, $w_j \Subset w_{i'}$.
(B) Now, suppose that $i \neq i'$. Without loss of generality, suppose that $\hat{m}' \leq m'$.
  (a) If $p - m - dl < p - n - \hat{d}l'$, then $w_j \Subset w_{i'}$.
  (b) (i) If $p - m - dl = p - n - \hat{d}l'$ and $|w_i| < |w_{i'}|$, then $w_i \Subset w_{i'}$.
    (ii) If $p - m - dl = p - n - \hat{d}l'$ and $|w_{i'}| < |w_i|$, then $w_{i'} \Subset w_i$.
  (c) Suppose that $p - m - dl > p - n - \hat{d}l'$ and $p - n - \hat{d}l' > m' - 1$.
    (i) If $p - n - \hat{d}l' + |w_{i'}| \leq p - m - dl + |w_i|$, then $w_{i'} \Subset w_j w_i$.
    (ii) Otherwise, $w_i \Subset w_{i'}$.
  (d) Suppose that $p - m - dl > p - n - \hat{d}l'$ and $p - n - \hat{d}l' < m' - 1$.
    (i) If $p - n - \hat{d}l' + |w_{i'}| \geq p - m - dl$, then $w_j \Subset w_{i'}$.
    (ii) If $p - n - \hat{d}l' + |w_{i'}| < p - m - dl$ and $p - n - \hat{d}l' + 2|w_{i'}| \leq p - m - dl + |w_i|$, then $w_{i'} \Subset w_j w_i$.
    (iii) Otherwise, $w_i \Subset w_{i'}$.
  (e) Suppose that $p - m - dl > p - n - \hat{d}l'$ and $p - n - \hat{d}l' = m' - 1$.
    (i) If $w_{i'} = w_j$ and $|w_i| < |w_{i'}|$, then $w_i \Subset w_{i'}$.
    (ii) If $w_{i'} = w_j$ and $|w_{i'}| < |w_i|$, then $w_{i'} \Subset w_i$.
    (iii) If $w_{i'} \neq w_j$ and $|w_{i'}| < |w_j|$, then $w_{i'} \Subset w_j$.
    (iii) If $w_{i'} \neq w_j$ and $|w_j| < |w_{i'}|$, then $w_j \Subset w_{i'}$.

In all cases, we contradict (S2). So, ϕ is well defined. The inverse is simply a one-block code.

Step 4

The essential conjugacy relabels every $w \in W''$. Let $\Omega' = \phi(W')$ (the set of words that are the relabeling of some $w \in W'$). For $0 \leq i \leq K + N$, let $\Omega_i = \phi(v_i)$. Recall that Y is generated by $\{v_0, v_1, \ldots, v_{K-1}\} \cup \{v_K, v_{K+1}, \ldots, v_{K+N}\}$. Then Z is a renewal system generated by $\{\Omega_0, \Omega_1, \ldots, \Omega_{K-1}\} \cup \{\Omega_K, \Omega_{K+1}, \ldots, \Omega_{K+N}\}$. Also, every word in $\{\Omega_K, \Omega_{K+1}, \ldots, \Omega_{K+N}\}$ is a finite concatenation of words from Ω'.

Consider the renewal system $\bar{Z}$ which is generated by $\Omega'' = \Omega' \cup \{\Omega_0, \Omega_1, \ldots, \Omega_{K-1}\}$. Every symbol in the alphabet of $\bar{Z}$ occurs in at most one word in Ω'' and at most once in any word. So, $\bar{Z}$ is a shift of finite type.

Since Z is generated by $\{\Omega_0, \Omega_1, \ldots, \Omega_{K-1}\} \cup \{\Omega_K, \Omega_{K+1}, \ldots, \Omega_{K+N}\}$, $\bar{Z}$ is generated by $\Omega' \cup \{\Omega_0, \Omega_1, \ldots, \Omega_{K-1}\}$, and every word in $\{\Omega_K, \Omega_{K+1}, \ldots, \Omega_{K+N}\}$ is a finite concatenation of words from Ω', Z can be described by the set of forbidden words in $\bar{Z}$ and a set of forbidden finite concatenations of words from Ω'. Let F denote this set of forbidden finite concatenations of words in Ω'. We need to show that F can be defined by some finite set of forbidden words.

Let $u \in F$. $\Omega' = \phi(W')$. So, u is forbidden in Z if and only if $\phi^{-1}(u)$ is forbidden in Y. A finite concatenation of words from W' is forbidden in Y if and only if that finite concatenation has been relabeled by ϕ_i for some i, where $1 \leq i \leq K$. So, u is forbidden in Z if and only if for some positive integer α and for some j, where $0 \leq j \leq K - 1$, $\phi^{-1}(u) = [\phi_{j+1}(w_j)]^{\alpha}$. Therefore, Z can be described by a finite set of forbidden words, and Z is a shift of finite type.

Since ϕ is an essential conjugacy and, for $1 \leq i \leq K$, ϕ_i is a conjugacy, X is essentially conjugate to a shift of finite type. □

Our methods require that the renewal system have a bounded generated set. This does not mean that Theorem 4.5 cannot be extended by other methods. Exploring this possibility would be a natural next step. One might also hope for a theorem similar to Theorem 4.5 for a more general class of sofic shifts.

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